Lecture 4: Nonparametric Methods

- Wilcoxon Signed Rank Test
- Wilcoxon Rank Sum Test
- Parametric vs nonparametric tests

Nonparametric Test

- Assumptions in the t-test
  - normal distribution of the underlying population
  - or large sample size
- If the assumptions are not satisfied, the t-statistic does not follow a t-distribution closely.

  - Parametric tests – based on the assumption that the shape of the population is known
  - Nonparametric tests – fewer restriction on the underlying distribution of data; do not rely on the Central Limit Theorem
    - Uses ranks rather than actual values

Nonparametric Test

- Sometimes transforming the data (e.g., log) can make the data look more normal.

  - A nonparametric test is robust to non-normality. This means the results are valid even when the distributions are highly non-normal.
    - Less sensitive to measurement error
    - Less sensitive to outliers
    - But loss of information

- Nonparametric tests is on the population median, not population mean
- Median is robust to outliers

Nonparametric Test

- So if there is a method that makes fewer assumptions, why not use that all the time?
- If the data really do follow normal distribution, there is loss of power!
  - That means if $H_0$ is false, a larger sample size would be needed to provide sufficient evidence to reject it

  - Steps are the same as before:
    - Specify the null and alternative hypotheses
    - Select a random sample of observations
    - Calculate a test statistic
    - Based on the value of the test statistic, either reject or do not reject $H_0$
Wilcoxon Signed-Rank Test

• A nonparametric version of the paired t-test
• Like the paired t-test, this test focuses on the difference in values for each pair of observations
• Example: Respiratory rates in premature infants
  – A study was conducted to examine the effects of the transition from fetal to postnatal circulation in premature infants
  – Respiratory rate was measured for each infant at two different times: <15 days old and >25 days old
  – We wish to compare the respiratory rates measured at two different times in the infants’ development

Wilcoxon Signed-Rank Test

• Let $\Delta$ be the true median difference in measurements
• $H_0: \Delta = 0$ vs $H_A: \Delta \neq 0$
• We first calculate the difference for each pair

<table>
<thead>
<tr>
<th>$x_{1i}$</th>
<th>$x_{2i}$</th>
<th>diff</th>
<th>$x_{1i}$</th>
<th>$x_{2i}$</th>
<th>diff</th>
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<td>-1</td>
<td>68</td>
<td>45</td>
<td>-23</td>
</tr>
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<td>38</td>
<td>40</td>
<td>2</td>
<td>67</td>
<td>31</td>
<td>-36</td>
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<td>80</td>
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<td>-38</td>
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<td>70</td>
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<td>-35</td>
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</tbody>
</table>

Wilcoxon Signed-Rank Test

• We rank the absolute values of the differences from smallest to largest
• Pairs for which the difference is 0 are not ranked
  – These pairs provide no information about which group has higher or lower values
  – Exclude these pairs and adjust the sample size accordingly
• Tied observations are assigned an average rank

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<th>rank</th>
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<tr>
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<tr>
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<td>2</td>
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<tr>
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<td>3</td>
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<td>13</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
</tr>
</tbody>
</table>

Sign Test:
Simply compare the number of positive signs to the expected number under $H_0$

Wilcoxon Signed-Rank Test

• Each rank is assigned a plus or a minus sign, depending on the sign of the difference
• Under the null hypothesis, the sum of the positive ranks should be comparable in magnitude to the sum of the negative ranks
• The statistic: the sum of the positive ranks, denoted by $T$
• We reject $H_0$ if $T$ is either too big or too small
• If $H_0$ is true and sample size is large,

$$Z_T = \frac{T - \mu_T}{\sigma_T}$$

follows a N(0,1) distribution

$$\mu_T = \frac{n(n+1)}{4}$$

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{24}}$$
Example

- The formula are slightly different when there are ties
- Back to the study on respiratory rates for premature infants (n=18)
- The sum of positive ranks:
  \[ T = 2+3+5+8+9 \]
- Under \( H_0 \), \( \mu_T = \frac{n(n+1)}{4} = 85.5 \)
- \( \text{Var}(T) = \frac{n(n+1)(2n+1)}{24} - C = 527.125 \)
- \( Z_T = \frac{(27-85.5)}{\sqrt{527.125}} = -2.55 \)
  - Since \( Z_T < -1.96 \), we reject \( H_0 \) and conclude that the median difference is not equal to 0.

Wilcoxon Rank Sum Test

- Analogous to the two-sample unpaired \( t \)-test
- Also called Mann-Whitney Test
- The null hypothesis being tested is that the median of the first population is equal to the median of the second population
- Like the signed-rank test, the rank sum test is performed on ranks rather than actual measurements

Example

- We test the supplement’s effect on SBP by giving supplement to one sample and placebo to an independent sample.
- Data:
  - Sample 1 (supplement): 118, 148, 110, 120, 140
  - Sample 2 (placebo): 135, 116, 120, 135, 145
- 1) combine samples into one group and order them
  - Ordered values: 110, 116, 118, 120, 135, 140, 145, 148
- 2) Assign ranks from lowest to highest; assign average rank to tied values
  - Ranks: 1, 2, 3, 4.5, 4.5, 6, 7, 8, 9
- 3) Compute statistic
  - \( W = 2 + 4.5 + 6 + 8 = 20.5 \)
More Groups

• You may wish to comparing three or more populations without assuming that the underlying data are not normally distributed

• The Kruskal-Wallis Test

• Paired t-test → Wilcoxon Signed Rank Test
• Unpaired t-test → Wilcoxon Rank Sum Test
• ANOVA → Kruskal-Wallis Test

R programming

wilcox.test(x, y = NULL, alternative = c("two.sided", "less", "greater"), mu = 0, paired = FALSE, exact = NULL, correct = TRUE, conf.int = FALSE, conf.level = 0.95, ...)